# A Logic-Based Representation for Coalitional Games with Externalities

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## ABSTRACT

We consider the issue of representing coalitional games in multi-agent systems that exhibit *externalities from coalition formation*, i.e., systems in which the gain from forming a coalition may be affected by the formation of other co-existing coalitions. Although externalities play a key role in many real-life situations, very little attention has been given to this issue in the multi-agent system literature, especially with regard to the computational aspects involved. To this end, we propose a new representation which, in the spirit of Ieong and Shoham [9], is based on Boolean expressions. The idea behind our representation is to construct much richer expressions that allow for capturing externalities induced upon coalitions. We show that the new representation is *fully expressive*, at least as *concise* as the conventional partition function game representation and, for many games, *exponentially more concise*. We evaluate the *efficiency* of our new representation by considering the problem of computing the *Extended and Generalized Shapley value*, a powerful extension of the conventional Shapley value to games with externalities. We show that by using our new representation, the Extended and Generalized Shapley value, which has not been studied in the computer science literature to date, can be computed in time linear in the size of the input.

## Categories and Subject Descriptors

C.0 [**Computer Systems Organization**]: General

## General Terms

Algorithms, Economics, Languages

## Keywords

Coalition Formation, Partition Function Games, Shapley Value

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## 1. INTRODUCTION

Coalitional games have proved highly influential in multi-agent systems research as they capture opportunities for cooperation by explicitly modelling the ability of the agents to take joint actions as primitives [19]. Sandholm *et al.* distinguish three main research issues in the use of coalitional games in multi-agent systems [24]:

- **Coalition structure generation:** Finding a *coalition structure*, i.e., an exhaustive set of mutually disjoint coalitions, so that the performance of the entire system is optimized; and
- **Teamwork:** Optimizing the performance of each individual coalition;
- **Payoff division:** Dividing the gains from cooperation among agents so as to meet certain positive/normative criteria.

With respect to the last issue, a number of solutions have been proposed in the literature. Two of the best-known ones are the *Shapley value* and the *core* [19]. Both concepts concern the problem of dividing the gain from forming the *grand coalition*, i.e., the coalition containing all the agents in the system. Specifically, a division of payoffs to individual agents is in the core if, with this division, no coalition could be better off by deviating from the grand coalition. However, there are no guarantees that such a division exists and, even if it does, there are no guarantees that it will be "fair", where by fairness we mean, among others, that identical agents obtain the same payoff, the agents that do not make any contributions obtain zero payoff, and that entire wealth is distributed. On the other hand, Shapley showed that there exists a unique division of the grand coalition's payoff that meets these important "fairness" conditions [25]. This division is called the Shapley value.

Most research effort in the multi-agent systems literature concerning the Shapley value focused on games with *no externalities*, i.e., games where the performance of a coalition remains unchanged, regardless of how the other agents in the system are partitioned. However, this assumption does not always hold. In many situations, a newly created coalition can have a significant effect (either positive or negative) on the performance of other coalitions. These effects are called *externalities from coalition formation*.[1]

Externalities from coalition formation play an important role in many real life problems, and have been extensively studied in eco-

[1]See [15] for a discussion of alternative concepts of externalities in multi-agent systems.

nomics and marketing [2]. This issue has also been recently considered in the multi-agent context [15, 22, 14]. Intuitively, externalities are evident in any situation, where the utility of an agent or a coalition of agents depends on choices made by other agents in the system. For example, if a coalition of agents in a market with automated trading mechanisms decides to adopt a new trading strategy, it may increase its competitive edge against other traders.

From a computational point of view, one of the key issues in developing efficient solutions to coalitional games is the way a game is represented [29]. A straightforward listing of the values for all possible coalitions requires a space of exponential size. In contrast, a well-crafted representation may be able to exploit the structure of the system and, therefore, model it in a much more concise manner. This can also facilitate significantly more efficient solutions to challenging computational problems [6, 9, 15]. While many alternative representations have been proposed in the computer science/AI literature for games with no externalities [6, 3, 4, 9, 18, 1], very little attention has been paid in this regard to games with externalities.[2] A class of representations was recently proposed by Michalak et al. [15], with the aim of enabling an efficient solution of the coalition structure generation problem. However, this class is not scalable for larger systems since they require, in the best case, defining at least $2^n$ values for $n$ agents.

Against this background, in this paper:

- We develop a logic-based representation for coalitional games with externalities. This representation builds upon Ieong and Shoham's *marginal contribution nets* (MC-nets) representation of games with no externalities, which has proven to be very efficient with respect to a number of important computational problems [9, 17]. Specifically, in our representation, we consider a much richer structure of logical rules, that allows to capture externalities;
- We prove that the new representation is fully expressive, i.e., it is able to represent any coalitional game with externalities and is not restricted to any particular subclasses of these games;
- We show that, for many games, our representation is exponentially more concise compared to all available alternatives for games with externalities, namely the conventional *partition function game* representation [11], and the representations recently introduced by Michalak et al. [15]; and
- We show that it can be used to efficiently compute the Shapley value for games with externalities. This challenging problem is studied in the computer science literature for the first time and, building upon our new representation, we present two approaches to compute it. In both of them, the value is obtained in time linear in the number of logical rules in our representation.

The remainder of the paper is organized as follows. Section 2 introduces our basic notation and background. We describe our representation and evaluate its expressivity and conciseness in Section 3. In Section 4 we discuss the concept of Shapley value for games with externalities. In Section 5, we show how to compute this value in time which is linear in the size of the input. Conclusions follow.

## 2. NOTATION AND BACKGROUND

Let $N = \{a_1, \dots, a_n\}$ be a set of agents – the players of the game. A *characteristic function* $v$ is a mapping $v : 2^N \to \mathbb{R}$, i.e., it assigns to every coalition $C \subseteq N$ a real number $v(C)$, representing the value of $C$.[3]

**Example 2.1 (Characteristic function)**
*For $N = \{a_1, a_2, a_3\}$, a sample characteristic function is $v(\{a_1\}) = 0$, $v(\{a_2\}) = 0$, $v(\{a_3\}) = 1$, $v(\{a_1a_2\}) = 1$, $v(\{a_1a_3\}) = 1$, $v(\{a_2a_3\}) = 1$ and $v(\{a_1a_2a_3\}) = 2$.*

A *game in a characteristic function form* is a tuple $(N, v)$, with components $N$ and $v$ as defined above. However, for ease of notation we will denote it by $v$ alone. A *coalition structure*, denoted $\pi$, is a disjoint and exhaustive partition of the agents in $N$. We denote by $\Pi(N)$ the space of all coalition structures over $N$.

**Example 2.2 (Coalition structures)**
*For $N = \{a_1, a_2, a_3\}$, the set $\Pi(N)$ contains $\{\{a_1\}\{a_2\}\{a_3\}\}$, $\{\{a_1a_2\}\{a_3\}\}$, $\{\{a_1a_3\}\{a_2\}\}$, $\{\{a_1\}\{a_2a_3\}\}$, and $\{\{a_1a_2a_3\}\}$. Let us denote these coalition structures by $\pi_1, \dots, \pi_5$, respectively.*

An *embedded coalition* is a pair $(C, \pi)$, where $\pi \in \Pi(N)$ and $C \in \pi$. We let $M$ denote the set of all embedded coalitions, that is, $M := \{(C, \pi) : \pi \in \Pi(N), C \in \pi\}$. A *partition function* is a mapping $w : M \to \mathbb{R}$. A *game in a partition function form* is a tuple $(N, w)$. Again, for ease of notation we will denote it by $w$ alone. Our shorthand notation for the partition function is demonstrated in the following example:

**Example 2.3 (Partition function)**
*A sample partition function for $N = \{a_1, a_2, a_3\}$ is denoted as $\{\{a_1, 0\}\{a_2, 0\}\{a_3, 1\}\}$, $\{\{a_1a_2, 1\}\{a_3, 2\}\}$, $\{\{a_1a_3, 1\}\{a_2, 0\}\}$, $\{\{a_1, 0\}\{a_2a_3, 1\}\}$ and $\{\{a_1a_2a_3, 2\}\}$. In words, this means that in $\{\{a_1\}\{a_2\}\{a_3\}\}$ coalitions $\{a_1\}$ and $\{a_2\}$ have value 0, whereas $\{a_3\}$ has value 1, etc.*

The game defined in Example 2.3, as opposed to Example 2.1, has externalities. In particular, the value of $\{a_3\}$ in $\pi_1$ is 1 whereas in $\pi_2$ it is 2. This means, that the formation of coalition $\{a_1a_2\}$ induced a positive externality of 1 on $\{a_3\}$.

As a more concrete example, let us consider coalition formation among four autonomous agent-robots $r_1$, $r_2$, $r_3$ and $r_4$. Each agent is able to secure the payoff of 1 if acting alone. No pair of cooperating agents achieves any value added. In fact, due to various incompatibilities, agents $r_1$ and $r_4$ do not perform well whenever they cooperate with each other decreasing their joint value or a value of any coalition they both belong to by $-0.5$. Their joint performance declines even further (by an additional $-0.5$) whenever they face cooperation of two other agents $r_2$ and $r_3$. As far as triplets are concerned, the coalition of the first three agents, i.e. $\{r_1, r_2, r_3\}$, is able to achieve an additional payoff of 1. When $r_4$ joins them to form the grand coalition the payoff increases even further by 0.5. The partition function for this game is as follows:

**Example 2.4 (Partition function for Cournot oligopoly)**

| | |
|---|---|
| $\{\{r_1, 1\}\{r_2, 1\}\{r_3, 1\}\{r_4, 1\}\}$ | |
| $\{\{r_1r_2, 2\}\{r_3, 1\}\{r_4, 1\}\}$ | $\{\{r_1r_2r_4, 2.5\}\{r_3, 1\}\}$ |
| $\{\{r_1r_3, 2\}\{r_2, 1\}\{r_4, 1\}\}$ | $\{\{r_1r_3r_4, 2.5\}\{r_2, 1\}\}$ |
| $\{\{r_1r_4, 1.5\}\{r_2, 1\}\{r_3, 1\}\}$ | $\{\{r_1, 1\}\{r_2r_3r_4, 3\}\}$ |
| $\{\{r_1, 1\}\{r_2r_3, 2\}\{r_4, 1\}\}$ | $\{\{r_1r_2, 2\}\{r_3r_4, 2\}\}$ |
| $\{\{r_1, 1\}\{r_2r_4, 2\}\{r_3, 1\}\}$ | $\{\{r_1r_3, 2\}\{r_2r_4, 2\}\}$ |
| $\{\{r_1, 1\}\{r_2, 1\}\{r_3r_4, 2\}\}$ | $\{\{r_1r_4, 1\}\{r_2r_3, 2\}\}$ |
| $\{\{r_1, r_2, r_3, 4\}\{r_4, 1\}\}$ | $\{\{r_1r_2r_3r_4, 5\}\}$ |

Clearly, the above game cannot be represented with a characteristic function. For instance, the value of coalition $\{r_1r_2\}$ is different in $\{\{r_1r_2\}\{r_3\}\{r_4\}\}$ than in $\{\{r_1r_2\}\{r_3r_4\}\}$. Thus, the

[2]See [29, Section 6] for a brief but informative overview of the main approaches to games with no externalities.

[3]To save space, where there is no risk of confusion, we will omit commas when listing sets, for example writing $\{\{a_1a_2\}\{a_3\}\}$ as a shorthand for $\{\{a_1, a_2\}, \{a_3\}\}$.

partition function is needed. Generally, the set of all partition function games is denoted as $W$. This includes all coalitional games that can be represented with a characteristic function, known as *c-games* [26]. More formally, a game $w \in W$ can be classified as a c-game if there exist a characteristic function game $v$ such that $\forall \pi, C$ such that $C \in \pi$, $w(C, \pi) := v(C)$. In this case, as in [13], we will call $v$ the *corresponding game in characteristic function form to* $w$. The set of c-games will be denoted as $W^c$.

In the remainder of this section, we will formally introduce the Shapley value. It was originally proposed as a normative method for dividing the value of the grand coalition among all of the agents in c-games [25]. Technically, an agent's share of a grand coalition's payoff is the average marginal contribution of that agent over all possible permutations of the agents in the system. Formally, using the notation that $|S|$ is the number of elements in a set $S$:

**Definition 2.5 ((standard) Shapley value)**
*For all $w \in W^c$, the Shapley value $Sh_i(v)$ for an agent $a_i \in N$ is given by:*

$$Sh_i(v) := \sum_{i \in C \subseteq N} \frac{(|C|-1)!(|N|-|C|)!}{|N|!} \Big( v(C) - v(C \backslash \{i\}) \Big),$$

*where $v$ is the game corresponding to $w$ in characteristic function form.*

Shapley showed that this is a unique value, that satisfies all of the following "fairness" axioms [25], where $\{N\}$ denotes the grand coalition:

- **Efficiency** $\sum_{i \in N} Sh_i(v) = v(\{N\})$;
- **Symmetry** $\forall C \subseteq N \backslash \{i, j\}$: $v(C \cup \{i\}) = v(C \cup \{j\})$ then $Sh_i(v) = Sh_j(v)$;
- **Null-Player** if $j$ is a null player in $v$ then $Sh_j(v) = 0$;
- **Linearity** (i) $\forall_{i \in N}$ $Sh_i(v + v') = Sh_i(v) + Sh_i(v')$, (ii) $\forall_{i \in N, \gamma \in \mathbb{R}}$ $Sh_i(\gamma v) = \gamma Sh_i(v)$.

Let us consider the following example:

**Example 2.6 (Shapley value for c-games)**
*For the c-game $v$ defined in Example 2.1, the Shapley values of every agent are $Sh_1(v) = \frac{1}{2}$, $Sh_2(v) = \frac{1}{2}$ and $Sh_3(v) = 1$.*

## 3. EMBEDDED MC-NETS

In this section we introduce our representation for games with externalities and evaluate its properties. We call this representation *embedded MC-nets*. A natural starting point for developing our representation is to consider c-games for which a number of representations have already been studied in the literature [6, 3, 4, 9, 18]. In particular, we build upon Ieong and Shoham's logic-based MC-nets representation for c-games [9] due to its desirable properties, i.e., it is fully expressive, concise for many c-games, and facilitates a very efficient way of computing the Shapley value. In MC-nets a c-game is represented with a set of rules $\mathcal{R}$, each rule of the form:

$$Pattern \longrightarrow Value,$$

where $Pattern$ is a Boolean expression over $N$. A coalition $C$ is said to *meet* a given pattern $\mathcal{P}$ if $\mathcal{P}$ evaluates to `true` when the values of all Boolean variables that correspond to agents in $C$ are set to `true`, and the values of all Boolean variables that correspond to agents not in $C$ are set to `false`. In this case, we write $C \models \mathcal{P}$. In MC-nets, the value of a coalition $C$ is the sum of all $Values$ from the rules of which the $Patterns$ are met by $C$. More formally:

$$v(C) = \sum_{\mathcal{R} \ni \mathcal{P} \rightarrow Value : C \models \mathcal{P}} Value.$$

Ieong and Shoham showed that from the MC-net representation of a game, which is limited to rules made only of conjunctions of positive and/or negative literals, the Shapley value can be computed in time linear in the size of the input [9, Section 4]. Following [7], we will call this class of rules *basic rules* and the representation based on them *basic MC-nets*. Using similar notation to that in formula (4), any basic rule can be written as:

$$p \wedge \neg p' \rightarrow Value, \qquad (1)$$

where $p$ and $p'$ are conjunctions of positive and negative literals, respectively. Now, due to linearity of the Shapley value, every basic rule can be considered as a separate game. Assuming that $p$ contains at least one positive literal, the Shapley values for any agent $a_i$ in $p$ and $a_j$ in $\neg p'$ can be computed from the following formulas:

$$\frac{Value}{|p| \binom{|p|+|\neg p'|}{|\neg p'|}} \quad \text{and} \quad \frac{-Value}{|\neg p'| \binom{|p|+|\neg p'|}{|p|}} \qquad (2)$$

respectively, where $|p|$ ($|\neg p'|$) denotes the number of elements in $p$ ($\neg p'$).

**Box 1:** Ieong and Shoham's method for computing Shapley value

If $C$ does not meet any rule then its value is 0. Every rule in the MC-nets is to be interpreted as a *marginal contribution* of agents to any coalition they jointly belong to. Since marginal contributions constitute the core element of the Shapley value definition, the MC-nets are particularly efficient in computing this solution concept (see Box 1 for more details on the work of Ieong and Shoham in this respect).

**Example 3.1 (MC-nets representation)**
*The c-game in Example 2.1 can be represented with just two rules: $a_3 \rightarrow 1$ and $a_1 \wedge a_2 \rightarrow 1$. Coalitions $\{a_1\}$ and $\{a_2\}$ do not meet any rules. Coalitions $\{a_3\}$, $\{a_1, a_3\}$ and $\{a_2, a_3\}$ meet the first rule, whereas coalition $\{a_1, a_2\}$ meets the second rule. The grand coalition meets both rules.*

Coalitional games with externalities are more complex mathematical objects than c-games, since we must represent the value not just of individual coalitions, but of embedded coalitions. Our approach is to extend such rules so that the $Pattern$ on the left hand side of a rule matches against coalition structures and not just coalitions. Specifically, our rules have the form:

$$\mathcal{P}_0 | \mathcal{P}_1, \ldots, \mathcal{P}_k \rightarrow \textit{Value}, \qquad (3)$$

where each individual pattern $\mathcal{P}_i : i \in \{0, \ldots, k\}$ is a Boolean expression over $N$ as defined for MC-nets. It should be noted that there is no obligation to specify any of the elements of the rule except for "$\rightarrow$" and "*Value*". For example, the following rules are all feasible:[4]

$a_1 \wedge a_2 \wedge \neg a_3 | a_3 \wedge a_4, a_5 \wedge \neg a_6 \rightarrow 1;$
$\emptyset | a_1 \wedge \neg a_2 \rightarrow 5;$
$a_1 \rightarrow -2.$

Within any pattern, if an agent is preceded with the sign "$\neg$", it will be referred to as a negative literal. Otherwise, it will be called a positive literal. The rule in (3) will be called an *embedded rule* and the entire left hand side of this rule will be called an *embedded pattern*, denoted $\mathcal{EP}$. A coalitional game with externalities can be represented by a tuple $(N, \mathcal{ER})$, where $\mathcal{ER}$ is a finite set of embedded rules. As mentioned at the beginning of this section, we call this representation *embedded MC-nets*.

An embedded coalition $(C, \pi)$ is said to *meet* an embedded pattern $\mathcal{EP}$ defined in (3), which we denote by $(C_0, \pi) \models \mathcal{EP}$, if:

- $C$ meets pattern $\mathcal{P}_0$; and
- every pattern $\mathcal{P}_j$, $j = 1, \ldots, k$ is met by at least one coalition in $\pi \setminus C$.

[4]For simplicity of notation, we assume that the empty set is met by any coalition.

In the special case of (3) being of the form $\mathcal{P}_0 = \emptyset$, the embedded rule is met by all the embedded coalitions $(C, \pi)$ for which $\pi \backslash C$ meets the requirement of $\mathcal{P}_1, \ldots, \mathcal{P}_k$. Similarly, if (3) is of the form $\mathcal{P}_0 \rightarrow$ *Value* then the embedded rule is met by all the embedded coalitions $(C, \pi)$ for which $C$ meets $\mathcal{P}_0$.

Embedded MC-nets, as defined in (3), allow for arbitrary Boolean expressions within the embedded pattern. However, similarly to the original MC-nets, our computational results will be derived for a special case of this representation, where the patterns in all rules are required to be conjunctions of literals (see Box 1 for more details on the work of Ieong and Shoham in this respect). Thus, when convenient, we will denote each pattern $\mathcal{P}_i$ $(i = 0, \ldots, k)$ from (3) as a conjunction of positive and negative literals $p_i \wedge \neg p'_i$, respectively. The embedded pattern (3) in the general form can be denoted as:

$$p_0 \wedge \neg p'_0 | p_1 \wedge \neg p'_1, \ldots, p_k \wedge \neg p'_k \rightarrow Value \qquad (4)$$

For example, in the rule $a_1 \wedge a_2 \wedge \neg a_3 | a_3 \wedge a_4, a_5 \wedge \neg a_6 \rightarrow 1$, we have $p_0 = a_1 \wedge a_2$, $\neg p'_0 = \neg a_3$, $p_1 = a_3 \wedge a_4$, $p_2 = a_5$ and $\neg p'_2 = \neg a_6$.

Furthermore, we will denote by $P_i$, $P'_i$ the sets containing agents in $p_i$, $\neg p'_i$, respectively. For example, in the above rule we have $P_0 = \{a_1 a_2\}$, $P'_0 = \{a_3\}$, $P_1 = \{a_3 a_4\}$, $P_2 = \{a_5\}$ and $P'_2 = \{a_6\}$. Additionally, the following assumption is imposed on any embedded rule:

$$|P_0| + \sum_{i=1}^{k} |P'_i| \geq 1, \qquad (5)$$

which follows an implicit assumption in [9] that any rule has at least one positive literal.

The value $w(C, \pi)$ of an embedded coalition $(C, \pi)$ is defined as the sum of all *Values* from the embedded rules that are met by $(C, \pi)$. More formally:

$$w(C, \pi) = \sum_{\mathcal{ER} \ni \mathcal{EP} \rightarrow Value : (C,\pi) \models \mathcal{EP}} Value.$$

**Example 3.2 (Embedded MC-nets)**
*The coalitional game with externalities from Example 2.3 can be described with the following set of rules:* $a_3 \rightarrow 1$, $a_3 | a_1 \wedge a_2 \rightarrow 1$ *and* $a_1 \wedge a_2 \rightarrow 1$.

The first two rules in the above example highlight the difference between games with and without externalities. According to the first rule, agent $a_3$ contributes to any embedded coalition it belongs with value 1. However, additionally, the second rule says that the value of this agent increases by 1 if there co-exists another embedded coalition in which agents $a_1$ and $a_2$ cooperate. This happens in $\pi_2 = \{\{a_1 a_2\}\{a_3\}\}$. In this way, embedded MC-nets allow us to capture externalities in coalitional games.

Now, when we have formally defined our representation, we will evaluate its properties. We start with expressiveness:

**Proposition 3.3 (Expressiveness)**
*Every coalitional game with externalities that is represented with a partition function can be expressed as embedded MC-nets.*

PROOF. To prove this, we will demonstrate that, for any arbitrary partition function $w$, there exists a set of rules $\mathcal{ER}$ such that for all $(C, \pi) \in M$ there exist a unique embedded rule $\mathcal{EP} \rightarrow Value \in \mathcal{ER}$ such that $(C, \pi) \models \mathcal{EP}$ and $Value = w(C, \pi)$. Thus, for any $(C, \pi)$:

$$\sum_{\mathcal{ER} \ni \mathcal{EP} \rightarrow Value : (C,\pi) \models \mathcal{EP}} Value = w(C, \pi).$$

This means that $(N, \mathcal{ER})$ represents the game $w$.

Let $(C_0, \pi)$ be an embedded coalition with $\pi = \{C_0, C_1, \ldots, C_m\}$. Let us consider the following embedded pattern:

$$p_0 \wedge \neg \bigcup_{i \neq 0} p_i | p_1 \wedge \neg \bigcup_{i \neq 1} p_i,\ p_2 \wedge \neg \bigcup_{i \neq 2} p_i, \cdots,\ p_m \wedge \neg \bigcup_{i \neq m} p_i, \qquad (6)$$

where $p_i$ is a conjunction of the agents in $C_i$. Every pattern $p_j \wedge \neg \bigcup_{i \neq j} p_i$, $j = 0, \ldots, m$ is met only by the embedded coalition $(C_j, \pi)$ and, therefore, the embedded pattern (6) is met only by the embedded coalition $(C_0, \pi)$. □

**Corollary 3.4 (Conciseness)**
*Embedded MC-nets are at least as concise as the partition function game representation.*

**Proposition 3.5 (Conciseness w.r.t. certain games)**
*Embedded MC-nets are exponentially more concise than the partition function game representation for certain games.*

PROOF. The above proposition follows easily from well-known results in Boolean function theory: essentially, a set of Boolean formulas provides a representation that in many cases is exponentially more succinct than the extensive representation (as the partition function game representation considered here), but in the worst case we need an exponential number of such formulas [28]. □

**Example 3.6**
*Let us consider an embedded MC-net representation of the game defined in Example 2.4. Specifically, only eight rules are needed to represent this game as embedded MC-nets, namely:* $r_1 \rightarrow 1$, $r_2 \rightarrow 1$, $r_3 \rightarrow 1$, $r_4 \rightarrow 1$, $r_1 \wedge r_4 \rightarrow -0.5$, $r_1 \wedge r_4 | r_2 \wedge r_3 \rightarrow -0.5$ *and* $r_1 \wedge r_2 \wedge r_3 \rightarrow 1$.

It should be noted that the above proposition hold also when we compare the embedded MC-nets to the representations introduced in Michalak et al. [15]. Specifically, embedded MC-nets are exponentially more concise than these representations for certain games.

Finally, it is easy to see that:

**Observation 3.7 (Generalization of MC-nets)**
*Embedded MC-nets are the generalization of MC-nets to games with externalities.*

With the above propositions and the corollary we have demonstrated the power of the embedded MC-net representation with respect to expressivity and conciseness. In the next two sections we will discuss its computational properties with respect to calculating the Shapley value in games with externalities.

# 4. SHAPLEY VALUE FOR GAMES WITH EXTERNALITIES

Since Shapley's original work, many refinements and modifications to his value have been proposed (see, e.g., [23]). Two of these extensions are directly related to games with externalities:

- One research direction focused on the *problem of generalization*, i.e., defining a value which can be computed for any *a priori* known coalition structure and not just the grand coalition. However, this so called *Generalised Shapley value* is still confined to the c-games domain;
- In another direction, authors tried to extend the Shapley value to games with externalities. Several solutions to this *problem of extension* have been proposed, each based on different assumptions. These solutions are called Extended Shapley values.

Now, whereas the generalization of the Shapley value is widely accepted, this is not yet the case for the Extended Shapley values. In this context, McQuillin has recently proposed a Shapley value for games with externalities that encompasses characteristics of both Generalized and Extended Shapley values [13]. In particular, he showed that the widely accepted solution to the problem of generalization forces a unique solution to the problem of extension, which he calls the *Extended, Generalized Shapley value* (EGSV). This value is not only unique but has a powerful property: namely, that all the other extended values considered in the literature *asymptotically converge* to it.

Before we formalize this concept, for any set $T \subseteq \pi$, let us define $\lfloor T \rfloor := \bigcup_{A \in T} A$. For instance, if $T = \{\{a_1\}\{a_2\}\} \subseteq \pi_1$ then $\lfloor T \rfloor = \{a_1 a_2\}$. Now, EGSV is defined as follows [13]:

**Definition 4.1 (Extended, Generalised Shapley value)**
*Given $w \in W$ and $(C, \pi) \in M$, the EGSV of $w$, denoted by, $\chi(w)$ is a game in partition function form defined as:*

$$\chi(w)(C,\pi) := \sum_{C \in T \subseteq \pi} \frac{(|T|-1)!(|\pi|-|T|)!}{|\pi|!} \Big( v(\lfloor T \rfloor) - v(\lfloor T \backslash \{C\} \rfloor) \Big)$$

*, where $v(S) := w(S, \{S, N \backslash S\})$.*

In words, to compute the EGSV of an embedded coalition $(C, \pi)$, one should construct a game $w_\pi$ in a characteristic function form, of which **the players are the coalitions** from the *a priori* coalition structure $\pi$ and the payoffs are given by: $w_\pi(T) := w\Big(\lfloor T \rfloor, \{\lfloor T \rfloor, \lfloor \pi \backslash T \rfloor\}\Big)$ for all $T \subseteq \pi$. Now, by computing in the game $w_\pi$ the conventional Shapley value for player $C$, we obtain $\chi(w)(C, \pi)$. Figure 1 depicts the process of computing the EGSVs for a coalitional game $w$ represented with a partition function. There are four agents in this game, $N = \{a_1, a_2, a_3, a_4\}$, and the *a priori* coalition structure is $\pi = \{\{a_1\}\{a_2\}\{a_3 a_4\}\}$.

It should be underlined that EGSVs can be computed for all $(C, \pi)$ in game $w$. These values create themselves a game in partition function form, which we will denote $\chi(w)$. The following example demonstrates how to compute $\chi(w)$ for the game considered in Example 2.3:

**Example 4.2 (EGSVs computed from the partition function)**
*Let us compute EGSVs for a priori coalition structure $\pi_2$ in the game with externalities from Example 2.3. We construct a c-game $w_{\pi_2}$ with two players $\{a_1 a_2\}$ and $\{a_3\}$. The characteristic function for this game is obtained from $w$ in the following way:*

$w_{\pi_2}(\{\{a_1 a_2\}\}) = w(\{\{a_1 a_2\}\}, \pi_2) = 1;$
$w_{\pi_2}(\{\{a_3\}\}) = w(\{\{a_3\}\}, \pi_2) = 2;$
$w_{\pi_2}(\{\{a_1 a_2\}, \{a_3\}\}) = w(\{\{a_1 a_2 a_3\}\}, \pi_5) = 2.$

*The EGSVs for coalitions in $\pi_2$ in game $w$ are conventional Shapley values computed from game $w_{\pi_2}$. These are:*

$\chi(w)(\{a_1 a_2\}, \pi_2) = Sh_{\{a_1 a_2\}}(w_{\pi_2}) = \frac{1}{2};$
$\chi(w)(\{a_3\}, \pi_2) = Sh_{a_3}(w_{\pi_2}) = \frac{3}{2}.$

*Similar computations for the remaining coalition structures, i.e., $\pi_1$, $\pi_3$, $\pi_4$ and $\pi_5$, yield:*

$\chi(w)(\{a_1\}, \pi_1) = \frac{1}{3}, \qquad \chi(w)(\{a_2\}, \pi_3) = \frac{1}{2};$
$\chi(w)(\{a_2\}, \pi_1) = \frac{1}{3} \qquad \chi(w)(\{a_2, a_3\}, \pi_4) = \frac{3}{2};$
$\chi(w)(\{a_3\}, \pi_1) = \frac{4}{3} \qquad \chi(w)(\{a_1\}, \pi_4) = \frac{1}{2};$
$\chi(w)(\{a_1, a_3\}, \pi_3) = \frac{3}{2} \qquad \chi(w)(\{a_1, a_2, a_3\}, \pi_5) = 2.$

*It is easy to see that $\chi(w)$ is a game in partition function form and it can be presented using the notion from Example 2.3 as:*

$\{\{a_1, \frac{1}{3}\}\{a_2, \frac{1}{3}\}\{a_3, \frac{4}{3}\}\} \qquad \{\{a_1, \frac{1}{2}\}\{a_2 a_3, \frac{3}{2}\}\}$
$\{\{a_1 a_2, \frac{1}{2}\}\{a_3, \frac{3}{2}\}\} \qquad \{\{a_1 a_2 a_3, 2\}\}$
$\{\{a_1 a_3, \frac{3}{2}\}\{a_2, \frac{1}{2}\}\}$

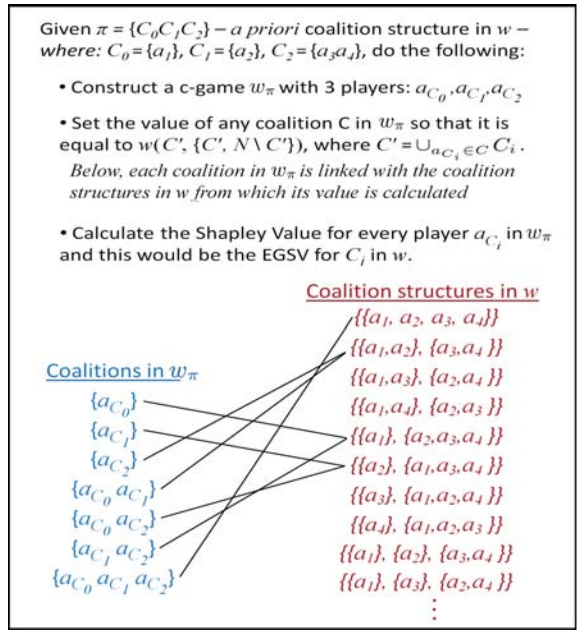


**Figure 1: An example of computing the EGSV**

McQuillin [13] shows that, in addition to efficiency, symmetry, null-player, linearity, the EGSV meets the following set of additional axioms: weak monotonicity, the rule of generalisation, strong linearity, cohesion, generalised null-player and recursion. We will take a closer look at two of them.[5] Specifically, strong form of linearity is enforced by the rule of generalisation and the linearity axiom and is defined as:

$$\chi(\gamma w + \gamma' w')(C, \pi) = \gamma \chi(w) + \gamma' \chi(w'), \ \gamma, \gamma' \in \mathbb{R} \qquad (7)$$

This follows from the simple observation that the operation $w \to \lambda w$ for a given $\lambda \in \Lambda_\pi$ is linear.

The recursion axiom is fundamental to McQuillin's extension:

$$\chi(\chi(w)) = \chi(w).$$

The basic meaning of this axiom is that the solution should be stable, i.e., the EGSVs of any game should be the EGSVs of themselves. The intuition behind this requirement is as follows. If the $\chi(w)$ is to be considered as a solution, then applying solution to the solution, i.e., $\chi(\chi(w))$ should not alter it.

**Example 4.3 (Recursion of $\chi(w)$)**
*It is easy to check that computing the EGSVs for the $\chi(w)$ from Example 4.2 results in $\chi(w)$.*

McQuillin proves very interesting result concerning recursion axiom. This result relates other concepts of Shapley value for games with externalities to his approach:

**Theorem 4.4 (McQuillin [13, Theorem 4])**
*Let $\chi$ be a solution which satisfies: efficiency, symmetry, linearity, cohesion, and generalised null–player or equivalently: efficiency, symmetry, null–player, linearity, and the rule of generalisation and if the following condition is fulfilled:*

$$\forall_{(C,\pi) \in M, |\pi| > 2} \quad \chi(w^{\beta}_{(C,\pi)})(C, \pi) > -\frac{|\pi| - 1}{|\pi|}$$

[5] For more details on all the axioms see [13].

*then the sequence* $\{\chi_t\}_{t=1}^{\infty}$*, where* $\chi_1 = \chi, \chi_2 = \chi \circ \chi, \chi_3 = \chi \circ \chi \circ \chi, \ldots$ *converges to the EGSV.*

The extended values proposed by [5, 12, 16, 20, 21] satisfy the requirements of Theorem 4.4 and, thus, after the iteration procedure, converge to the EGSV. Finally, McQuillin shows that his value is the only possible value that meets all the axioms considered:

**Theorem 4.5 (McQuillin [13, Theorem 2])**
*Axioms efficiency, symmetry, linearity, weak monotonicity, cohesion, generalised null-player and recursion or equivalently: efficiency, symmetry, null–player, linearity, weak monotonicity, the rule of generalisation and recursion are satisfied by* $\chi$ *if and only if* $\chi$ *is the EGSV.*

In the next section we will evaluate the efficiency of embedded MC-nets with respect to computing EGSVs.

# 5. COMPUTING EGSV WITH EMBEDDED MC-NETS

As mentioned in Section 3, our computational results are derived for a special case of the embedded MC-net representation that has only positive and/or negative literals in the embedded rules. Specifically, we will present two algorithms for computing EGSVs for an *a priori* coalition structure $\pi$ in a game with externalities $w$ that is represented with embedded MC-nets. Both algorithms are illustrated in Figure 2 and briefly described below:

(a) As we know from Section 4, EGSVs are computed by constructing a c-game $w_\pi$ with players being the coalitions in $\pi$ and computing the (conventional) Shapley values for this game. These values are equal to the EGSVs for $\pi$ in game $w$ (i.e. $\chi(w)(C, \pi)$). Having this in mind, we develop a linear-time procedure for transforming the embedded MC-net representation of $w$ to the basic MC-net representation of $w_\pi$ using the same number of rules. This will allow us to take advantage of the fact that the Shapley value can be computed using the basic MC-nets in time linear in the number of rules. Now, since the transformation procedure and the Shapley value calculation both require a linear number of operations, it follows that our proposed algorithm of calculating $\chi(w)(C, \pi)$ is also linear. The scheme of this algorithm is illustrated in Figure 2(a) and described in more detail in Section 5.1;

(b) In the second algorithm, depicted in Figure 2(b) and described in Section 5.2, we transform $w$ into $\chi(w)$. This transformation also takes time linear in the number of embedded rules and $\chi(w)$ is made of twice as much embedded rules as $w$. The game $\chi(w)$ can then be used to compute EGSVs for any particular $\pi$. In contrast to algorithm (a), the transformation in algorithm (b) is done only once. After that, for any *a priori* coalition structure, the EGSVs can be be obtained straight away from $\chi(w)$. On the other hand, algorithm (b) requires storing twice as many embedded rules compared to algorithm (a).

Before we introduce both algorithms we need the following lemma and definition. The lemma directly follows from the strong linearity axiom (7) that is met by the EGSV:

**Lemma 5.1**
*Let* $w$ *be a game represented with embedded MC-nets* $(N, \mathcal{ER})$*. Furthermore, let* $w_{r_i}$ *be the game represented with only one embedded rule* $rule_i \in \mathcal{ER}$*. The EGSVs for* $w$*, i.e.* $\chi(w)$*, are equal to the sum of the EGSVs computed for every* $w_{rule_i}$*, where* $rule_i \in \mathcal{ER}$*.*

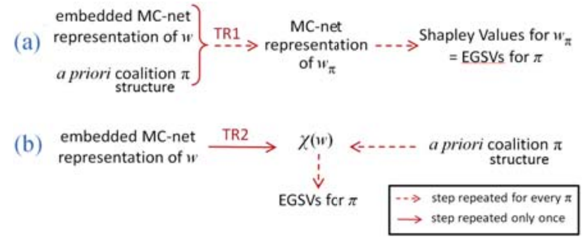


**Figure 2: Algorithms of computing EGSVs from our embedded MC-net representation. TR1 and TR2 denote the first and the second transformation procedures, respectively.**

The above lemma means, in particular, that, in what follows, we can focus on a single, representative embedded rule instead of a set of such rules. All results concerning derivation of the EGSVs for a game described by a single rule extend to any game described by a set of rules.

**Definition 5.2 (Division by a coalition structure)**
*Let* $\pi$ *be a coalition structure made of agents in* $N$*. We will say that* $\pi$ ***divides*** *two subsets* $A$ *and* $B$ *of* $N$ *if there exists* $T \subset \pi$ *such that* $A \subseteq \lfloor T \rfloor$ *and* $B \subseteq N \backslash \lfloor T \rfloor$*.*

For instance, $\pi = \{\{a_1 a_2 a_3\}\{a_4 a_5\}\}$ divides $A = \{a_1 a_3\}$ and $B = \{a_4\}$. Clearly, for two subsets $A$ and $B$ to be divided by a partition $\pi$ it is necessary to be disjoint.

## 5.1 Algorithm computing EGSV for an *a priori* coalition structure

Let $w$ be a game with externalities given by a single embedded rule as defined in (4). Furthermore, let $\pi = \{C_0, \ldots, C_m\}$ be the *a priori* coalition structure and let us denote by $N_\pi = \{a_{C_1}, \ldots, a_{C_m}\}$ a set of players in game $w_\pi$. Now, we propose the following procedure to transform the embedded rule (4) that describes the game $w$ into a basic rule that describes the game $w_\pi$:

Step 1 (a) if the embedded rule is of the form $p_0 \rightarrow Value$, i.e. it is a basic rule with only positive literals, then the transformed rule is:

$$\mathbf{p}_0 \rightarrow Value, \tag{8}$$

where $\mathbf{p}_0$ denotes a conjunction of all $a_C \in N_\pi : C \cap P_0 \neq \emptyset$.

(b) Otherwise go to Step 2.

Step 2 (a) if $\pi$ divides $P_0 \cup \bigcup_{i>0} P'_i$ and $P'_0 \cup \bigcup_{i>1} P_i$ then we transform (4) into the following basic rule:

$$\mathbf{p} \wedge \neg\mathbf{p}' \rightarrow Value, \tag{9}$$

where $\mathbf{p}$ and $\neg\mathbf{p}'$ are conjunctions of agents from $N_\pi$ with only positive and only negative literals, respectively, and $\mathbf{p}$ contains agents $a_C \in N_\pi : C \cap (P_0 \cup \bigcup_{i \geq 1} P'_i) \neq \emptyset$ and $\neg\mathbf{p}'$ contains agents $a_C \in N_\pi : C \cap (P'_0 \cup \bigcup_{i \geq 1} P_i) \neq \emptyset$.

(b) otherwise the transformed rule is $\emptyset \rightarrow 0$.

**Theorem 5.3 (Computing $w_\pi$ from embedded MC-nets)**
*Given game* $w$ *described by a single embedded rule (4) and an a priori coalition structure* $\pi$*, the transformed rule produced by the above procedure describes the c-game* $w_\pi$*.*

PROOF. The values of the c-game $w_\pi$ are assigned from the coalition structures in game $w$ that contain no more than two coalitions (see Section 4 for more details). Based on this, our procedure involves identifying, and then transforming, the rules that are met by the coalitions embedded in these coalition structures. Now:

Step 1. If the rule describing game $w$ is a basic rule with only positive literals then it will always be met by at least the grand coalition. Thus, it should be transformed and included in the basic MC-nets representation of $w_\pi$. This transformation goes as follows. Since players in $w_\pi$ are coalitions from $\pi$, we replace every agent $a_i$ in expression $p_0$ with the player $a_C \in N_\pi$ such that $a_i \in C \cap P_0 \neq \emptyset$. Intuitively, since every coalition $(C,\pi)$ is always considered in $w_\pi$ as a single player $a_C$, then even if a single agent $a_i$ from $C$ appears in $p_0$, the contribution of $a_C$ is the same as the contribution of $a_i$. Hence, $a_C$ appears in $\mathbf{p}_0$.

Step 2. Recall that we are interested in embedded rules met by $(C,\pi) : \pi \leq 2$, i.e. $\pi$ is either of the form $\{C, C'\}$ or it is the grand coalition. If $\pi$ divides sets $P_0 \cup \bigcup_{i>0} P'_i$ and $P'_0 \cup \bigcup_{i>1} P_i$ then, following the definition of the embedded MC-nets, agents in these sets will belong to two disjoint coalitions. This means that the embedded rule (4) will apply to coalition structures of the form $\{C, C'\}$. Similarly to Step 1, we replace agents from $N$ that are in either $P_0 \cup \bigcup_{i>0} P'_i$ and $P'_0 \cup \bigcup_{i>1} P_i$ with agents from $N_\pi$. The division requirement ensures that this replacement can always be done. Furthermore, the form of the basic rule $\mathbf{p} \wedge \neg\mathbf{p'} \to Value$ guarantees that $Value$ will be assigned to only those coalitions that contain all the agents in $\mathbf{p}$ and non of the agents in $\mathbf{p'}$. In other words, the meaning of the basic rule (9) corresponds to the meaning of the embedded rule (4). $\square$

**Example 5.4 (Computing $w_\pi$ from embedded MC-nets)**
*Let us consider game $w$ described with the three embedded rules in Example 3.2 and let $\pi_2$ be an a priori coalition structure. The first and the third rule are basic rules. Following Definition 5.2, $\pi_2$ divides the second rule which is an embedded rule. Having in mind that $N_\pi = \{a_{C_1} a_{C_2}\}$, where $a_{C_1} = \{a_1 a_2\}$ and $a_{C_2} = \{a_3\}$, these three rules, due to Theorem 5.3, should be transformed into basic rules describing $w_{\pi_2}$ as follows:*

$$a_3 \to 1 \quad \Rightarrow \quad a_{C_2} \to 1;$$
$$a_3 | a_1 \wedge a_2 \to 1 \quad \Rightarrow \quad a_{C_2} \wedge \neg a_{C_1} \to 1;$$
$$a_1 \wedge a_2 \to 1 \quad \Rightarrow \quad a_{C_1} \to 1.$$

Now, we can compute $\chi(w)(\{a_1a_2\}, \pi_2)$ and $\chi(w)(\{a_3\}, \pi_2)$ from the basic rules describing $w_{\pi_2}$. This is done by applying the linear method proposed by Ieong and Shoham (see Box 1) which yields the following values: $Sh_{a_{C_1}}(w_{\pi_2}) = \frac{1}{2}$ and $Sh_{a_{C_2}}(w_{\pi_2}) = \frac{3}{2}$. These Shapley values are the EGSVs for $(C_1, \pi_2)$ and $(C_2, \pi_2)$; thus $\chi(w)(\{a_1a_2\}, \pi_2) = \frac{1}{2}$ and $\chi(w)(\{a_3\}, \pi_2) = \frac{3}{2}$. See how these values match those computed from the partition function game representation in Example 4.2. This completes our first algorithm.

## 5.2 Algorithm computing the game of EGSVs

To compute $\chi(w)$ from game $w$ that is represented with embedded MC-nets, we will need a new type of patterns which has the following form:

$$a_{i_1} \vee \cdots \vee a_{i_k} | \pi \text{ divides } A \text{ and } B, \tag{10}$$

where $A$ and $B$ are disjoint sets of agents. An embedded coalition $(C,\pi)$ meets the requirement of a rule with this pattern if $\pi$ divides $A$ and $B$ and at least one of the agents $a_{i_1}, \ldots, a_{i_k}$ belongs to $C$. If $C_1 = \{a_{j_1}, \ldots, a_{j_l}\}$ is a coalition of agents then $C_1^\vee$ will denote the pattern $a_{j_1} \vee \cdots \vee a_{j_l}$, if $C_2 = \{a_{h_1}, \ldots, a_{h_m}\}$ is another coalition of agents then $C_1^\vee \vee C_2^\vee := \{a_{j_1} \vee \cdots \vee a_{j_l} \vee a_{h_1} \vee \cdots \vee a_{h_m}\}$. We use the convention that the pattern $\emptyset^\vee$ is not met by any coalition.

Now, we propose our second transformation procedure that transforms embedded rules describing $w$ into rules of the type (10) describing $\chi(w)$. Let $w$ be game with externalities described with a single embedded rule (4). Game $\chi(w)$ is given by the following two exclusive rules:

$$\begin{aligned} &P_0^\vee \vee \bigvee_{i\geq 1} P_i'^\vee \mid \pi \text{ divides } P_0 \cup \bigcup_{i\geq 1} P'_i \text{ and } P'_0 \cup \bigcup_{i\geq 1} P_i \to V \cdot D(C,\pi), \\ &P_0'^\vee \vee \bigvee_{i\geq 1} P_i^\vee \mid \pi \text{ divides } P_0 \cup \bigcup_{i\geq 1} C'_i \text{ and } P'_0 \cup \bigcup_{i\geq 1} P_i \to V \cdot D(C,\pi)', \end{aligned} \tag{11}$$

where $V$ denotes $Value$ and coefficients $D$ and $D'$ are defined as follows:

$$\begin{aligned} D(C,\pi) &:= \frac{1}{r\binom{r+s}{r}}, \quad r := |\{C \in \pi : C \cap (P_0 \cup \bigcup_{i\geq 1} P'_i) \neq \emptyset\}| \\ D(C,\pi)' &:= \frac{-1}{s\binom{r+s}{s}}, \quad s := |\{C \in \pi : C \cap (P'_0 \cup \bigcup_{i\geq 1} P_i) \neq \emptyset\}| \end{aligned} \tag{12}$$

if both $r$ and $s$ are non zero. However, if $r = 0$ then $D(C,\pi) = 0$ and if $s = 0$ then we put $D(C,\pi)' = 0$. if both $r$ and $s$ are non zero. However, if $r = 0$ then $D(C,\pi) = 0$ and if $s = 0$ then we put $D(C,\pi)' = 0$.

**Theorem 5.5 (Computing $\chi(w)$ from embedded MC-nets)**
*If $w$ is given by a rule of the form (4) then $\chi(w)$ is given by two embedded rules of the form (11).*

PROOF. Let $(C,\pi)$ be an embedded coalition. From definition we know that the value game $\chi(w)(C,\pi)$ is the Shapley value of $a_C$ in the game $w_\pi$. If $w$ is given by the embedded rule (4) and $\pi$ divides $P_0 \cup \bigcup_{i\geq 1} P'_i$ and $P'_0 \cup \bigcup_{i\geq 1} P_i$ then $w_\pi$ is given by the rule (9), otherwise $w_\pi$ is given by $\emptyset \to 0$. Let us now assume that $\pi$ divides $P_0 \cup \bigcup_{i\geq 1} P'_i$ and $P'_0 \cup \bigcup_{i\geq 1} P_i$. Then $\chi(w)(C,\pi) = Sh_{a_C}(w_\pi)$ and, according to Ieong and Shoham (see Box 1), it is equal to $Value D(C,\pi)$ if $C \cap (P_0 \cup \bigcup_{i\geq 1} P'_i) \neq \emptyset$ or $Value D(C,\pi)'$ if $C \cap (P'_0 \cup \bigcup_{i\geq 1} P_i) \neq \emptyset$ or 0 otherwise.

The embedded coalition $(C,\pi)$ can match at least one of the rules (11). Then its value, according to (11) is equal to $Value D(C,\pi)$ or $Value D'(C,\pi)$ depending which of the exclusive rules (11) applies. Thus, it follows that $\chi(w)$ is given by the rule (11).

Let $w$ be a game represented by an embedded MC-net $(N, \mathcal{ER})$. Then the EGSVs of $w$ is equal to the sum of the EGSVs over the embedded rules (Lemma 5.1), and thus the game $\chi(w)$ can be represented by aggregating all the rules (11) transformed from all the rules in $\mathcal{ER}$. $\square$

**Example 5.6 (Computing $\chi(w)$ from embedded MC-nets)**
*Let us consider the same game and the a priori coalition structure as in the previous example. According to the Theorem 5.5 $\chi(w)$ can be represented by the following set of rules:*

$$\{a_3\} \to D_1(C,\pi);$$
$$\{a_3\} \mid \pi \text{ divides } \{a_1, a_2\} \text{ and } \{a_3\} \to D_2(C,\pi);$$
$$\{a_1 \vee a_2\} \mid \pi \text{ divides } \{a_1, a_2\} \text{ and } \{a_3\} \to D'_2(C,\pi);$$
$$\{a_1 \vee a_2\} \to D_3(C,\pi).$$

Now, using the transformed rules from Example 5.6, we will compute EGSVs for $\pi_2$. Coalition structure $\pi_2$ divides $\{a_1, a_2\}$ and $\{a_3\}$ thus all three rules will apply. We obtain:

$$\chi(w)(\{a_1,a_2\},\pi_2) = \overbrace{D_1(\{a_1,a_2\},\pi_2)}^{=0} + \overbrace{D_2(\{a_1,a_2\},\pi_2)}^{=0} + \underbrace{D_3(\{a_1,a_2\},\pi_2)}_{=\frac{1}{2}} = \frac{1}{2}$$

$$\chi(w)(\{a_3\},\pi_2) = \overbrace{D_1(\{a_3\},\pi_2)}^{=1} + \overbrace{D_2(\{a_3\},\pi_2)}^{=1}$$
$$\underbrace{D_2'(\{a_3\},\pi_2)}_{=-\frac{1}{2}} + \underbrace{D_3(\{a_3\},\pi_2)}_{=0} = \frac{3}{2}$$

# 6. CONCLUSIONS

In this paper we propose a logic-based representation for coalitional games with externalities, called embedded MC-nets. We demonstrate that it is fully expressive and at least as concise as the conventional partition function game representation. We also show that it can be exponentially more concise. This result also extends to the recently proposed representations of Michalak et al. [15]. We test the efficiency of the embedded MC-nets by considering the problem of computing the Extended, Generalized Shapley value, which adapts the Shapley notion to games with externalities. While the computation of this value always requires an exponential number of operations when the conventional partition function game representation is used, and this is due to the size of this representation; we demonstrate that it can be computed with our representation in time linear in the number of embedded rules. We propose two alternative algorithms to perform these computations.

There are two obvious extensions of our work. Firstly, in the spirit of Elkind at al. [7], the embedded rules with more complex Boolean expressions can be considered. Secondly, we are keen on testing the properties of the embedded MC-net representation with respect to another solution concepts of cooperative games (see [27] for an overview). The main of them include the core, already mentioned in the introduction, as well as the stable sets, the bargaining set, the kernel and the nucleolus. The computational properties of the bargaining set and the kernel for games represented in a concise manner have been recently studied in [8]. In general, similarly to the Shapley value and the nucleolus, the solution to this two concepts always exists. However, their specific feature is that a solution may contain multiple different divisions of payoff among agents [19]. Nevertheless, it should be underlined that the Shapley value is a normative solution concept, whereas all the other listed above are positive ones (as they refer to various notions of stability based on interests of individual agents). The first step of a potential analyse should be to extend notions of these positive solution concepts to coalitional games with externalities. In fact, as the example of the Shapley value shows, it may be a challenging task by itself. As far as the core is concerned, such extensions have been already considered in the literature (see [10] for a short but informative overview).

# 7. ACKNOWLEDGMENTS

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